Cauchy’s Integral Theorem II

Integrals in Multiply Connected Domains

Cauchy-Goursat Theorem
Cauchy’s Integral Theorem II

To reiterate, the Cauchy Integral Theorem states that in a simply connected domain $D$, the integral of an analytic function $f(z)$ on a simple closed contour is zero:

$$\oint_C f(z) \, dz = 0$$
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Things are a little different in a Multiply-connected domain.

(a) Simply connected domain

(b) Multiply connected domain

Two kinds of domains
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\[ \oint_C f(z) \, dz = (\text{who knows?}) \]

\[ \oint_C f(z) \, dz = 0 \]
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\[ \oint_{C_1} f(z) \, dz = (\text{who knows?}) \]

If we can figure out either integral, then we know the value of both!

\[ \oint_{C} f(z) \, dz = (\text{who knows? BUT ...}) \]

*We know it is the same as the value for the integral around \( C_1 \)*
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A further result of the Cauchy-Goursat theorem concerns triply and higher ranks of connected domains.

In a situation like the one pictured here, where a function \( f(z) \) is analytic everywhere in \( D \) except at the holes, then, because of the summing property of integrals we get:

\[
\int_C f(z) \, dz = \int_{C_1} f(z) \, dz + \int_{C_2} f(z) \, dz
\]
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In General...
If $C_1, C_2, C_3, \ldots, C_n$ are simple closed, non-overlapping curves, all with a positive orientation that all lie in the interior of the curve $C$, and if $f(z)$ is analytic at each point within $C$ but exterior to the all the $C_k$, then

$$\oint_C f(z) \, dz = \sum_{k=1}^{n} \oint_{C_k} f(z) \, dz$$
There is a special type of “hole” that often occurs in otherwise simple domains and that is when there is a simple zero in the denominator of the integrand.

We will find the value of a simple, general version of this type of integral and we will employ it many times in more specific problems.
Consider the function \( f(z) = \frac{1}{z-z_0} \). Find the integral of this function integrated on the contour \( C_1 \) at right.

Solution: Because the function is analytic everywhere on the complex plane except at \( z_0 \), we can use the principal of deformation to choose any path that completely encompasses \( z_0 \) and the results will have the same value.

We can pick any closed contour, so we will choose a simple circle with radius “\( r \)” centered on \( z_0 \). We will call it \( C_2 \). We will evaluate the integral using the 7-step method.
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\[ C_2 \Rightarrow \quad z(t) = re^{it} + z_0, \quad t: 0 \rightarrow 2\pi \]

\[ \dot{z}(t) = ire^{it} \]

\[ f(z) = \frac{1}{z - z_0} \]

\[ f(z(t)) = \frac{1}{z(t) - z_0} = \frac{1}{(re^{it} + z_0) - z_0} = \frac{1}{re^{it}} \]

\[ f(t) = \dot{z}(t)f(z(t)) = (ire^{it})\left(\frac{1}{re^{it}}\right) = i \]

\[ I = \int_{0}^{2\pi} (i)dt = 2\pi i \]
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\[ \oint_{C} \frac{dz}{z - z_0} = 2\pi i, \]

"C" is any positively oriented contour that contains \(z_0\)

There is no restriction on how small “\(r\)” can be as long as it is greater than zero. That means this result is **valid for any isolated simple pole**.
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\[ \oint_C \frac{dz}{z - z_0} = 2\pi i, \quad \text{"C" contains } z_0 \]

This is the **Cauchy-Goursat Theorem**

Please note well! ➔ The result is valid for a contour taken in the **COUNTERCLOCKWISE** sense. (That is the direction of positive angular change.)

Traversing the contour in the clockwise sense has the effect of swapping the limits of the integral and the result is \(-2\pi i\).
Example:

Evaluate \( \int_C \frac{dz}{z-i} \) for the contour \( C \) shown at right: The contour is made up of six segments, each of which individual parametric line integrals must be made up.
Instead, we can observe that

1) The function is analytic everywhere except at \( z = i \)
2) The pole of the function lies within the contour.

Therefore, by using the principal of deformation, we know we will get the same result by integrating on \( C_1 \) instead. The \( C_1 \) contour is a circle centered on \( z = i \) (counter-clockwise) and we know this integral evaluates to

\[
\int_C \frac{dz}{(z - i)} = \int_{C_1} \frac{dz}{(z - i)} = 2\pi i
\]
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Example:

Evaluate \( f(z) = \frac{z-13}{z^2+4z-5} \) integrated on the closed contour \(|z + 3 - 2i| = 3\) (clockwise).

The function is the quotient of two analytic functions, which means the function is analytic everywhere except where the denominator is zero. Analyze the denominator to find the roots of the denominator and, therefore, the poles of the function.
Example: Evaluate $f(z) = \frac{z-13}{z^2+4z-5}$ on $|z + 3 - 2i| = 3$ (clockwise).

First, determine the contour and sketch it.

The contour is a circle centered on $-3 + 2i$ with radius 3.
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Next, analyze the function to see if it is analytic within the boundaries of the contour.

The denominator can be factored:

\[ z^2 + 4z - 5 = (z + 5)(z - 1) \]

Break up the function \( \frac{z-13}{z^2+4z-5} \) into two simpler fractions.

\[ \frac{z - 13}{z^2 + 4z - 5} = \frac{z - 13}{(z + 5)(z - 1)} = \frac{A}{z + 5} + \frac{B}{z - 1} \]

(Yes, this is partial fraction expansion...)

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\[
\frac{z - 13}{z^2 + 4z - 5} = \frac{A}{z + 5} + \frac{B}{z - 1}
\]

for some $A$ and $B$ to be determined as follows:

By cross-multiplying the right hand side:

\[
\frac{A}{z + 5} + \frac{B}{z - 1} = \frac{(A + B)z - A + 5B}{(z + 5)(z - 1)}
\]

Compare the coefficient of $z$ and the constant term:

\[
A + B = 1 \quad \text{and} \quad -A + 5B = -13
\]

Solve this pair of linear equations in the usual way and get:

\[
A = 3 \quad \text{and} \quad B = -2
\]
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Substitute these values of \( A \) and \( B \):

\[
\frac{z-13}{z^2+4z-5} = \frac{3}{z+5} + \frac{-2}{z-1}
\]

Now re-write the integral:

\[
\oint_C \frac{z-13}{z^2+4z-5} \, dz = \oint_C \left[ \frac{3}{z+5} + \frac{-2}{z-1} \right] \, dz
\]

\[
= 3 \oint_C \frac{dz}{z+5} - 2 \oint_C \frac{dz}{z-1}
\]
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Now we need to take a good look at

\[ 3 \int_C \frac{dz}{z+5} - 2 \int_C \frac{dz}{z-1} \]

We look at the integrands to see if they have poles within the contour of integration. If not, then the integral is zero, (by the Cauchy integral theorem).

If an integrand **does** have a pole within the contour then the integral is \(2\pi i\) (by the Cauchy-Goursat theorem).

The first integral is non-analytic at \(z = -5\) and the second at \(z = +1\).
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$r = 3$

-5

+1
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\[ 3 \oint_C \frac{dz}{z+5} - 2 \oint_C \frac{dz}{z-1} \]

The pole of the first integrand falls within the contour of integration, so we invoke the Cauchy-Goursat Theorem, to wit:

\[ \oint_C \frac{dz}{z+5} = -2\pi i \text{ (negative because the contour is clockwise)} \]

There is a coefficient of 3, so we get

\[ 3 \oint_C \frac{dz}{z+5} = -6\pi i \]
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\[3 \oint_C \frac{dz}{z + 5} dz - 2 \oint_C \frac{dz}{z - 1}\]

The non-analytic point of the second integral (+1) falls outside of the contour of integration, so we can deal with it using the Cauchy Integral Theorem, like so:

\[2 \oint_C \frac{dz}{z - 1} = 0\]
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Putting the two parts back together and using our original expression, we can say:

$$\oint_C \frac{z - 13}{z^2 + 4z - 5} \, dz = -6\pi i$$

when integrated clockwise on the closed contour $|z + 3 - 2i| = 3$
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\[ f(x) = \frac{x^3 + 3x^2 - 5x + 2}{x^2 + 2x - 3} \]

**Partial Fraction Expansion Example**

Because the numerator has higher order than the denominator, start by dividing and finding the remainder.

\[ f(x) = x + 1 + \frac{-\frac{1}{4}x + 1}{x^2 + 2x - 3} \]

**Work on the Fraction:**

\[ \frac{-\frac{1}{4}x + 1}{x^2 + 2x - 3} = \frac{A}{x+3} + \frac{B}{x-1} = \frac{A(x-1) + B(x+3)}{(x-1)(x+3)} = \frac{Ax - A + Bx + 3B}{x^2 + 2x - 3} \]

\[ = \frac{(A+B)x - A + 3B}{x^2 + 2x - 3} \]

\[ \begin{align*}
A + B &= -\frac{1}{4} \\
-A + 3B &= 5
\end{align*} \]

\[ \Rightarrow \frac{AB}{4B} = 1 \]

\[ B = \frac{1}{4}, \quad A = -\frac{17}{4} \]

\[ f(x) = x + 1 + \frac{\frac{1}{4}}{x-1} - \frac{\frac{17}{4}}{x+3} \]
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**Example:**

**Evaluate:** \[ \oint_C \frac{5z + 8}{z^2 - z - 2} \, dz \]

where \( C = 10e^{it} \)

**0 \leq t \leq 2\pi**

**Break up the integrand using partial fraction expansion:**

\[
\frac{5z + 8}{z^2 - z - 2} = \frac{A}{z-2} + \frac{B}{z+1} = \frac{A(z+1) + B(z-2)}{(z-2)(z+1)}
\]

**Equate numerators:** \( (A+B) = 5 \quad A - 2B = 8 \)

**Solve for A & B:** \( A = 6, \quad B = -1 \)

**Substitute back into the original problem:**

\[
\oint_C \left( \frac{6}{z-2} - \frac{1}{z+1} \right) \, dz = \oint_C \frac{dz}{z-2} - \oint_C \frac{dz}{z+1}
\]

\( z = 2 \) \& \( z = -1 \) **are points of non-analyticity**

and they both lie within the contour of integration.

So, \( \oint_C \frac{dz}{z-2} = 2\pi i \) and \( \oint_C \frac{dz}{z+1} = 2\pi i \)

**And** \( \oint_C f(z) = (6 - 1) 2\pi i = 10\pi i \)