Rectangular and Polar Form

The Complex Plane

Recall that the complex plane is a two-dimensional arrangement of numbers using the real and imaginary number lines as axes.

The numbers on the axes are “pure” (real or imaginary).
Rectangular and Polar Form

The Complex Plane

The general complex number has been expressed as

\[ z = a + bi \]

This is referred to as “rectangular” or “Cartesian” form.

There is another, equally valid, notation for specifying the complex number \( z \): Polar Form.

Just as in analytic geometry, we can locate a point on a plane using radius (\( r \)) and angle (\( \theta \)) instead of \( x \) and \( y \) coordinates.
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\[ z = a + bi \]

As can be seen from the diagram, the values of \( a \) and \( b \) can be expressed in terms of \( r \) and \( \theta \) (using a little trig).

\[
\begin{align*}
    a &= r \cos \theta \\
    b &= r \sin \theta \\
    z &= (r \cos \theta) + i(r \sin \theta)
\end{align*}
\]

\[ z = r(\cos \theta + i \sin \theta) \]
**Rectangular and Polar Form**

\[ z = r(\cos \theta + i \sin \theta) \]

Also, as can be seen from the diagram, the values of \( r \) and \( \theta \) can be determined from the values of \( a \) and \( b \) (again using a little trig).

\[
\begin{align*}
r &= |z| = \sqrt{a^2 + b^2} \quad (always) \\
\theta &= \tan^{-1} \left( \frac{b}{a} \right) \quad (sometimes)
\end{align*}
\]

This formula for \( \theta \) is only valid for \(-90^\circ < \theta < +90^\circ\), that is, if \( \theta \) is in the 1\(^{st}\) or 4\(^{th}\) quadrants.
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\[ z = r(\cos \theta + i \sin \theta) \]

\[ \theta = \tan^{-1}\left(\frac{b}{a}\right) \quad (sometimes) \]

To make sure you have the correct value of \( \theta \), you must make sure \( \theta \) falls in the correct quadrant. Depending on the signs of \( a \) and \( b \), \( \theta \) may be more than 90° or less than -90°.
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\[ z = r(\cos \theta + i \sin \theta) \]

\[ \theta = \tan^{-1} \left( \frac{b}{a} \right) \] (sometimes)

To make sure you have the correct value of \( \theta \), do the following:

If \( a > 0 \) then \( \theta = \tan^{-1} \left( \frac{b}{a} \right) \)

If \( a < 0 \) then \( \theta = \tan^{-1} \left( \frac{b}{a} \right) \pm 180° \)

If \( a = 0 \), then

- if \( b > 0 \) then \( \theta = +90° \)
- else if \( b < 0 \) then \( \theta = -90° \)
- else \( b = 0 \), and therefore \( \theta \) is undefined.
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\[ z = r(\cos \theta + i \sin \theta) \]

Another, more compact notation is to use

\[ z = r \angle \theta \]

We will refer to this notation as “phasor notation” (although it must be kept in mind that \( z \) is not a phasor [it is a number]). This notation indicates that the number \( z \) has magnitude (or “modulus”) \( r \) and angle \( \theta \).
Graphical Representation:

As seen in the figure, a number $z$ can be located using a kind of vector drawing (although $z$ is not a vector [it is a number]).

This kind of representation can be used for graphical addition and subtraction.
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Graphical Representation:

An important distinction between polar form and rectangular form is that the angle is not unique. Whereas $z = a + bi$ is only true for unique values of $a$ and $b$, $z = r \angle \theta$ can also be correctly expressed as

$$z = r \angle (\theta + k \cdot 360^\circ), \quad k \in \mathbb{Z}$$

$$z = r \angle (\theta + k \cdot 2\pi), \quad k \in \mathbb{Z}$$

In many cases, all possible angle values must be considered.
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Graphical Representation:

The complex conjugate of $z$ in polar form is a simple matter of taking the negation of the angle while keeping the magnitude unchanged.

For $z = r \angle \theta$, $\bar{z} = z^* = r \angle (-\theta)$
Rectangular and Polar Form

Graphical Arithmetic:

As shown at right, the numbers $z_1$ and $z_2$ can be added graphically by treating them like vectors and repositioning them (“tip-to-tail”). The resultant “vector” indicates the point on the complex plane that is the number $z_3 = z_1 + z_2$.

This is the exact same result for adding the real parts of $z_1$ and $z_2$ to get the real part of $z_3$ and adding the imaginary parts of $z_1$ and $z_2$ to get the imaginary part of $z_3$. 
Graphical Arithmetic:

In similar manner, the number $z_2$ can be subtracted from $z_1$ graphically. The “vectors” are repositioned (“tip-to-tail”) but we use the negative of the $z_2$ “vector”. The resultant “vector” indicates the point on the complex plane that is the number $z_3 = z_1 - z_2$.

Again, this is what you get when you find the difference of the real parts to get the real part of $z_3$ and find the difference of the imaginary parts to get the imaginary part of $z_3$. 
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Graphical Arithmetic:

\( z_1 \) can be subtracted from \( z_2 \) in a similar manner.
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\[ z = r(\cos \theta + i \sin \theta) \]

**Polar notation** is particularly useful for multiplication and division. There are times when the numbers \( a \) and \( b \) are not so “nice” as they were in the examples above. An example will illustrate how this is done.

Given:

\[
z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)
\]

and

\[
z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)
\]

then

\[
z_1 z_2 = (r_1 r_2) [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]
\]
This kind of “polar-form multiplication” is even more compact when we use phasor notation.

Given: \( z_1 = r_1(\cos \theta_1 + i \sin \theta_1) = r_1 \angle \theta_1 \)
and \( z_2 = r_2(\cos \theta_2 + i \sin \theta_2) = r_2 \angle \theta_2 \)

then \( z_1z_2 = (r_1r_2) \angle (\theta_1 + \theta_2) \)

This can be proven using the definition of multiplication and conversion from rectangular to polar form.
Polar form really can be appreciated when doing division. Compare division in rectangular form and phasor notation:

Rectangular: \[
\frac{z_1}{z_2} = \frac{ac + bd + (bc - ad)i}{c^2 + d^2}
\]

Polar: \[
\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle (\theta_1 - \theta_2)
\]
Rectangular and Polar Form

So, here is the rule of thumb for doing simple arithmetic on complex numbers:

Addition and Subtraction: Use Rectangular Form

Multiplication and Division: Use Polar Form
• Graphical Multiplication
At this point the student might be wondering if there is an easy way to switch between polar and rectangular forms of a complex number.

Happily, there is a way…

On every scientific calculator worthy of the name, there are function keys to do this conversion.
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Even on the instructor’s ancient HP calculator, there are such keys:
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Somewhere on your calculator, you will probably find keys that look like:

→R  →P

Some calculators use different nomenclature such as “R2P” and “P2R”. These functions perform conversion from rectangular to polar and polar to rectangular.

These functions are very handy and it would behoove you to master them.
Exercise 1:

Given: \(x^2 - y^2 + ixy = 1 + ix\), and \(x, y \in \mathbb{R}\)

Find: pairs of \(x\) & \(y\) that satisfy the equation.

Move all terms to the left side of the equation
\[x^2 - y^2 + ixy - 1 - xi = 0\]

Separate real and imaginary parts
\[(x^2 - y^2 - 1) + i(xy - x) = 0\]

Real and imaginary parts must each be equal to zero. Set up separate equations.

\[x^2 - y^2 - 1 = 0\] \[1\]
\[xy - x = 0\] \[2\]

Solve [2] for possible values of \(x\) and \(y\).
\[x(y - 1) = 0 \implies x = 0\] \([3a]\) or \(y = 1\) \([3b]\)

\(If\ x = 0 \ then\ -y^2 - 1 = 0 \implies y = \pm i\)
So, \(if\ x = 0 \ then\ y\ is\ not\ real, \therefore x \neq 0\)
Exercise 1:
Given \(x^2 - y^2 + ixy = 1 + ix\), and \(x, y \in \mathbb{R}\) (Continued)

Substitute \(y = 1\) \([3b]\) into \(x^2 - y^2 - 1 = 0\) \([1]\) and analyze.

If \(y = 1\) then we get \(x^2 - (1)^2 - 1 = 0\)

Solve for \(x\)
\[x = \pm \sqrt{2}\]

These two values of \(x\) are real numbers and therefore valid.

\[
\begin{align*}
y &= 1, \\
x &= +\sqrt{2}
\end{align*}
\]
\[
\begin{align*}
y &= 1, \\
x &= -\sqrt{2}
\end{align*}
\]

\(\text{The solution set is made up of two } (x, y) \text{ pairs}\)
EXERCISE: WHAT IS THE CUBE ROOT OF 8?

(IN RECTANGULAR FORM,)

LET \( z = \sqrt[3]{8} \)

USE POLAR FORM

\[
\begin{align*}
\theta &= \text{non-neg real} \\
\rho &= \sqrt{8} \\
\end{align*}
\]

\[
z^3 = (\rho \angle \theta)^3 = 8
\]

\[
\rho^3 / 3 \theta = 8
\]

From the diagram, we can see:

\[
\rho^2 = 8, \quad 3 \theta = 0
\]

\[
:\rho = 2, \quad \theta = 0
\]

Is that it? Not quite...

Remember that in polar form, the angle is cyclical:

\[
z^3 = r^3 \angle 3\theta = 8 \angle 0^\circ + k \cdot 360^\circ, \quad k \in \mathbb{Z}
\]

Evaluate for all values of \( k \):

\[
\begin{align*}
3 \theta &= k \cdot 360^\circ \\
\theta &= k \cdot 120^\circ
\end{align*}
\]

| \( k \) | \( \theta \) | EQUIVALENT
|---|---|---|
| 0 | 0 | 0
| 1 | 120 | +120 \{ DISTINCT ANSWERS \}
| 2 | 240 | -120
| 3 | 360 | 0
| 4 | 540 | +120 \{ REPEATS \}

CUBE ROOTS OF 8
So, \( Z = 2e^{\theta^\circ}, \ 2e^{(120^\circ)}, \ 2e^{(-120^\circ)} \)

**CONVERT TO RECTANGULAR FORM:**

(This can be done using \( \rightarrow R \))

\[
\begin{align*}
2e^{0^\circ} &= 2(\cos(0) + i\sin(0)) = 2 + 0i = 2 \\
2e^{120^\circ} &= 2(\cos(120^\circ) + i\sin(120^\circ)) = 2(-0.5 + 0.866i) = -1 + 1.732i \\
2e^{-120^\circ} &= 2(\cos(-120^\circ) + i\sin(-120^\circ)) = 2(-0.5 - 0.866i) = -1 - 1.732i
\end{align*}
\]

**SOLUTION SET:**

\[
\sqrt[3]{8} = \left\{ \begin{array}{l}
2 \\
-1 + \sqrt{3}i \\
-1 - \sqrt{3}i
\end{array} \right. 
\]