As has been the pattern, functions of complex numbers and their derivatives are extensions of well-known functions from real-valued analysis. That is, a function takes input (one or more numbers), applies a math procedure and returns a result. In the math of complex numbers (“The Complex”), the same thing happens.

One of the differences is that it is often impossible fully visualize the function.

Consider the real-valued function $f(x) = x^2 + 3$

You should recognize that this is a parabola that opens upward:
\( f(x) = x^2 + 3 \)

The real-valued function is easy to visualize because the input (\( x \)) is one dimensional and the output is also one dimensional. Put them together and you get a two dimensional graph clearly showing the function.
This isn’t so easy for a function of \( z \). In the general case, the input is two dimensional and the output is two dimensional.

This means you have to imagine a four dimensional relationship to fully grasp the function.

So, most of our work will be with the algebra of the functions and only occasionally can we use diagrams.
Another way of thinking about $f(z)$ is as a mapping from one complex plane onto another. Note the labels on the axes.
While in general complex functions cannot be fully visualized, there are some complex functions that can be visualized if the output of the function is pure real or pure imaginary or if there is no phase information to convey.

Consider the function \( f(z) = \text{Re} \left( \frac{1}{z} \right) \). The output of this function is pure real values. The input is two dimensional and the output is one dimensional, so a 3-D graph can be produced.

We also observe that the function goes to zero as \( z \) tends to infinity in any direction. But it is interesting what happens when \( z \) tends toward zero...
\[ f(z) = \text{Re} \left( \frac{1}{z} \right) \]

In this view, the image is spun around from the usual to get a better view (the positive real goes to the left). The function is not defined at \( z = 0 \). The spikes actually extended up and down toward +/- infinity.
The formal definition of a complex function is:
A rule \( f \) that is defined on a set \( S \) and assigns a complex value \( w \) to every \( z \) in \( S \). The set \( S \) is called the **domain** of \( f \).

\[
w = f(z)
\]

In evaluating a complex function, it greatly helps to break down the function into its real and imaginary parts, both for the input of the function and the output.
In general, we let

\[ z = x + iy \]

where \( x \) and \( y \) are **real number variables** and they are the Cartesian values of the variable \( z \) on the complex plane. The output is generally expressed as

\[ f(z) = w = u(x, y) + iv(x, y) \]

where \( u \) and \( v \) are **real-valued functions** of the **real-valued variables** \( x \) and \( y \).

By **un-packing** the variables and functions this way, we are able to do conventional math on the components and re-assemble the complex numbers at the end.
For example: Given $f(z) = 3z^2 - z + 1$. Find $u(x, y)$ and $v(x, y)$ and $f(1 - 2i)$.

Start by replacing $z$ with $x + iy$ in the function definition and separate the real and imaginary parts:

**Substitute $x + iy$ for $z$**

$3z^2 - z + 1 = 3(x + iy)^2 - (x + iy) + 1$

**Expand algebraically**

$= 3x^2 + 6xyi - 3y^2 - x - iy + 1$

**Separate into real and imaginary parts**

$= (3x^2 - 3y^2 - x + 1) + i(6xy - y)$

So, $u(x, y) = 3x^2 - 3y^2 - x + 1$, and $v(x, y) = 6xy - y$
\( f(z) \): Functions and Derivatives

\[
f(z) = 3z^2 - z + 1
\]

\[
u(x, y) = 3x^2 - 3y^2 - x + 1, \text{ and } v(x, y) = 6xy - y
\]

For \( f(1 - 2i) \), \( x = 1 \) and \( y = -2 \). For our \( u \) and \( v \):

\[
u(1, -2) = 3 \cdot 1^2 - 3(-2)^2 - 1 + 1 = -9
\]

\[
v(1, -2) = 6 \cdot 1 \cdot (-2) - (-2) = -10
\]

\[
f(1 - 2i) = u(1, -2) + iv(1, -2) = -9 - 10i
\]
$f(z)$: Functions and Derivatives

Limit and Continuity of a Function:

A function is said to have a limit at a point $z_0$ if the function “goes to” some finite number $L$ as $z$ “goes to” $z_0$. This is written thus:

$$L = \lim_{{z \to z_0}} f(z)$$
$f(z)$: Functions and Derivatives

Limit and Continuity of a Function:

$$L = \lim_{z \to z_0} f(z)$$

This definition is the same as in real-valued analysis. In real number analysis, the limit must be the same if $x \to x_0$ from the positive or negative. In the complex, the limit must be the same as $z$ approaches $z_0$ from any direction.
As $z \to z_0$ along some path, $f(z) \to L$. If the limit of the function exists at $z_0$, then $f(z)$ will converge on $L$ **no matter which path is taken**. The definition of “limit” implies that $f(z)$ is defined in a neighborhood of $z_0$. 

(a) $\delta$-neighborhood  
(b) $\epsilon$-neighborhood
Limit and Continuity of a Function:

A function is **continuous** at \( z = z_0 \)

if \( f(z_0) \) is defined and \( \lim_{{z \to z_0}} f(z) = f(z_0) \)
Consider the function \( f(z) = \frac{e^z - 1}{z} \).
Clearly this function is not defined at \( z = 0 \).

But \( \lim_{z \to 0} f(z) \) does exist (can you find it?).

So, although the limit does exist at \( z = 0 \), it is discontinuous at that point.
Limit and Continuity of a Function:

If a function is continuous at every point of its domain then it is said to be “continuous in a domain”.
The Derivative of a Complex Function: $f'(z)$

The definition of a derivative of $f$ at the point $z_0$ in the complex is the extension of the definition for real valued analysis:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad \text{(if the limit exists)}. $$

The kicker is that $\Delta z \to 0$ \textit{from any direction}. This can also be written:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{if the limit exists.}$$

and $z \to z_0$ \textit{from any direction}. 
The Derivative of a Complex Function: $f'(z)$

example: for $f(z) = z^3 + 2$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^3 + 2 - (z^3 + 2)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z^3 + 3z^2 \Delta z + 3z(\Delta z)^2 + (\Delta z)^3 + 2 - z^3 - 2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(\Delta z)(3z^2 + 3z\Delta z + (\Delta z)^2)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (3z^2 + 3z\Delta z + (\Delta z)^2) = 3z^2$$
\( f(z) \): Functions and Derivatives

The Derivative of a Complex Function: \( f'(z) \)

So, \( f'(z) = (z^3 + 2)' = 3z^2 \) (just like in real valued analysis)

In fact, the differentiation rules are the same in the complex as for real valued analysis starting with the rules for simple exponentiation:

\[
[z^n]' = nz^{(n-1)} \quad \text{(as demonstrated above)}
\]

also (for constant \( c \)), \( (cf)' = c(f') \)

\( (f + g)' = f' + g' \)
\( f(z) \): Functions and Derivatives

The Derivative of a Complex Function: \( f'(z) \)

(Rules for differentiation continued):

\[
\begin{align*}
(z^n)' &= nz^{n-1} \\
(cf)' &= c(f') \\
(f + g)' &= f' + g' \\
(fg)' &= f'g + g'f \\
\left(\frac{f}{g}\right)' &= \frac{f'g - g'f}{g^2}
\end{align*}
\]

Also, the chain rule still works.
(Rules for differentiation continued):

Also the rule for $e^x$ carries over to complex analysis:

$$(e^z)' = e^z$$
\( f(z) \): Functions and Derivatives

The Derivative of a Complex Function: \( f'(z) \)

(Rules for differentiation continued):

Did we mention the chain rule? Remember the chain rule?

It goes like this: for functions \( f \) and \( g \),

\[
\frac{d(f(g))}{dz} = \frac{d(f(g))}{dg} \cdot \frac{dg}{dz}
\]

For example, for \( f(z) = e^z \) and \( g(z) = z^3 \)

\[
\left[ f(g(z)) \right]' = \left[ e^{z^3} \right]' = 3z^2 e^{z^3}
\]
There are many complex functions that do not have derivatives at any point. Consider: $f(z) = \text{Re}(z) + 5 \cdot \text{Im}(z) = x + 5y$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(x + \Delta x + 5y + 5\Delta y) - (x + 5y)}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta x + 5\Delta y}{\Delta x + i\Delta y}$$  \hspace{1cm} (Hmm ... ... now what?)
$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta x + 5\Delta y}{\Delta x + i\Delta y}$

$\Delta z = \Delta x + i\Delta y$, 
then for $\Delta z \to 0$, $\Delta x$ and $\Delta y$ must both go to zero.

$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta x + 5\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta x \to 0 \atop \Delta y \to 0} \frac{\Delta x + 5\Delta y}{\Delta x + i\Delta y}$
The Derivative of a Complex Function: $f'(z)$

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta x + 5\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \frac{\Delta x + 5\Delta y}{\Delta x + i\Delta y}$$

$\Delta x$ and $\Delta y$ must both go to zero but in either order.
The Derivative of a Complex Function: $f'(z)$

$$f'(z) = \lim_{\Delta x \to 0, \Delta y \to 0} \frac{\Delta x + 5\Delta y}{\Delta x + i\Delta y}$$

So, if we let $\Delta x \to 0$ first, we get

$$\lim_{\Delta y \to 0} \left( \lim_{\Delta x \to 0} \frac{\Delta x + 5\Delta y}{\Delta x + i\Delta y} \right) = \lim_{\Delta y \to 0} \frac{5\Delta y}{i\Delta y} = -5i$$

and if we let $\Delta y \to 0$ first, we get

$$\lim_{\Delta x \to 0} \left( \lim_{\Delta y \to 0} \frac{\Delta x + 5\Delta y}{\Delta x + i\Delta y} \right) = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

Because we get different results for different paths to $\Delta z = 0$, the limit does not exist. No limit? **No derivative.**